
KTUNOTES

Module-4

Electrostatic Energy and Energy density

Electric potential energy, or electrostatic potential energy is a potential energy (measured in Joules) that results from conservative Coulomb forces and is associated with the configuration of a particular set of point charges within a defined system.

or

Work done in forming a configuration of charges is called the electrostatic energy of the system. We assume that the charges were initially at infinity. The potential at a point gives the workdone in bringing a unit +ve charge from infinity to the point against the field. ∴ workdone in bringing a charge Q to the point where the potential is V is

$W = VQ$ joules

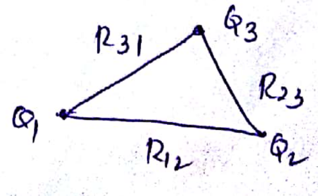
For two charges: Let Q_1 and Q_2 be the point charges separated by a distance R_{12}. Potential at site of Q_2 due to Q_1 is $\frac{1}{4\pi\varepsilon_0}\frac{Q_1}{R_{12}}$

∴ W.D in bringing a charge Q_2 is

$$W_E = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R_{12}}$$

For three charges

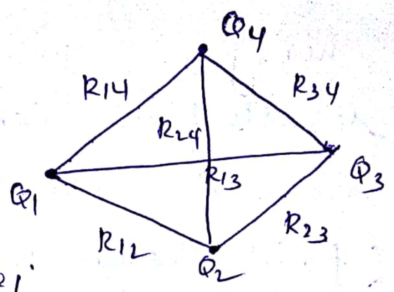
$$W_E = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1 Q_2}{R_{12}} + \frac{Q_2 Q_3}{R_{23}} + \frac{Q_3 Q_1}{R_{31}}\right]$$



Four charges

$$W_E = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1 Q_2}{R_{12}} + \frac{Q_2 Q_3}{R_{23}} + \frac{Q_3 Q_4}{R_{34}} + \frac{Q_4 Q_1}{R_{14}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_4}{R_{24}}\right]$$

$\therefore$ For N charges



Consider N point charges $Q_1, Q_2 \ldots Q_N$. Let R_{ij} be the separation between Q_i and Q_j.

Electrostatic potential of Q_i due to Q_j

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q_j}{R_{ij}}$$

W.D in bringing Q_i $= \frac{1}{4\pi\varepsilon_0}\frac{Q_i Q_j}{R_{ij}}$

Taking all possible values of $i, j = 1$ to N $(i \neq j)$ will lead to duplications like $\frac{Q_{ij}}{R_{ij}}$ and $\frac{Q_j Q_i}{R_{ji}}$

But these two terms are equal

The summation of W.D over all possible values of i and j $(i \neq j)$ gives duplication. Hence we divide the expression by 2.

$\therefore$ P.E of the system of N charges is

$$W_E = \frac{1}{2} \times \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{Q_i Q_j}{R_{ij}}, \quad i \neq j$$

$$= \frac{1}{2}\sum_{i=1}^{N} Q_i \sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{Q_j}{R_{ij}}$$

$$= \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$$

V_i is the potential of Q_i due to remaining $(N-1)$ charges.

...stead of point charges, the region has continuous charge distribution then

For line charge ρ_l $\quad W_E = \frac{1}{2}\int \rho_l \, dl \cdot V$ J

For surface charge ρ_s $\quad W_E = \frac{1}{2}\int \rho_s \, ds \, V$ J

For volume charge ρ_v $\quad W_E = \frac{1}{2}\int \rho_v \, dv \cdot V$ J

Energy density

Consider the volume charge distr. having ρ_v C/m³

$$W_E = \frac{1}{2}\int_v \rho_v V \, dv$$

According to Maxwell's equation (Gauss's law)

$$\rho_v = \nabla \cdot \bar{D}$$

$$\therefore W_E = \frac{1}{2}\int_v (\nabla \cdot D) V \, dv$$

For any vector A and scalar V the vector identity is

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A})$$

$$\therefore (\nabla \cdot \bar{A})V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V$$

$$\therefore W_E = \frac{1}{2}\int_v (\nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V) \, dv$$

$$= \frac{1}{2}\int_v (\nabla \cdot V\bar{D}) dv - \frac{1}{2}\int_v \bar{D} \cdot \nabla V \, dv$$

According to divergence theorem $\left(\oint_s A \cdot ds = \int_v \nabla \cdot A \, dv\right)$

$$\frac{1}{2}\int_v (\nabla \cdot V\bar{D}) dv = \frac{1}{2}\oint (V\bar{D}) \cdot d\bar{s}$$

$$\therefore W_E = \frac{1}{2}\oint (V\bar{D}) \cdot ds - \frac{1}{2}\int_v \bar{D} \cdot \nabla V \, dv \qquad E = -\nabla V$$

$$= \frac{1}{2}\oint (V\bar{D}) \cdot ds + \frac{1}{2}\int \bar{D} \cdot E \, dv$$

We know $V \propto \frac{1}{r}$ $D \propto \frac{1}{r^2}$ for point charges (DV

$V \propto \frac{1}{r^2}$, $D \propto \frac{1}{r^3}$ for dipole and so on

So $V\bar{D}$ is $\propto$ at least $\frac{1}{r^3}$ while ds varies as r^2

Hence total integral varies as $1/r$. As surface becomes very large $r \to \infty$ and $1/r \to 0$ Hence closed surface integral is zero

$$\therefore \; W_E = -\frac{1}{2}\int_V (\bar{D} \cdot \nabla V)\, dv$$

But $E = -\nabla V$

$$W_E = \frac{1}{2}\int_V \bar{D}\cdot(-\bar{E})\, dv$$

$$\therefore \; W_E = \frac{1}{2}\int_V (\bar{D}\cdot\bar{E})\, dv \;]$$

$D = \varepsilon_0 E$

$$W_E = \frac{1}{2}\int_V (\varepsilon_0 \bar{E}\cdot\bar{E})\, dv$$

$$W_E = \frac{1}{2}\int_V \varepsilon_0 E^2\, dv$$

$$\text{or } \; W_E = \frac{1}{2}\int_V \frac{D^2}{\varepsilon_0}\, dv \;]$$

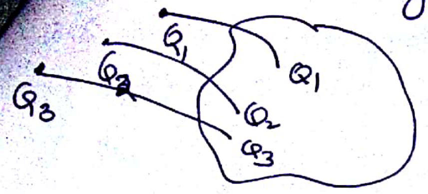
In differential form

$$dW_E = \frac{1}{2}\bar{D}\cdot\bar{E}\, dv$$

$$\frac{dW_E}{dv} = \frac{1}{2}\bar{D}\cdot\bar{E} \;\; J/m^3 \quad \text{(Energy density)}$$

$$W_E = \int_V \left(\frac{dW_E}{dv}\right) dv$$

Electrostatic energy



$$W = W_1 + W_2 + W_3$$

$$W = 0 + Q_2 V_{21} + Q_3 [V_{31} + V_{32}] \quad -(1)$$

If the order is changed i.e. Q_3 first

$$W = 0 + Q_2 V_{23} + Q_1 [V_{12} + V_{13}] \quad -(2)$$

$$2W = Q_1 [V_{12} + V_{13}] + Q_2 [V_{23} + V_{21}] + Q_3 [V_{31} + V_{32}]$$

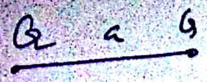
$$2W = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$W = \frac{1}{2} [Q_1 V_1 + Q_2 V_2 + Q_3 V_3]$$

$$W_E = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i$$

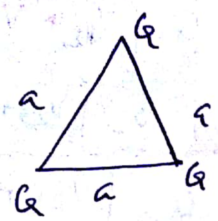
27) Calculate the electrostatic potential energy of the following systems of point charges. (i) 2 equal charges Q separated by distance 'a' (ii) Three equal charges Q placed at the vertices of an equilateral Δ of side 'a' (iii) A charges Q placed at the corners of a square of side 'a'. (iv) Six charges Q placed at the vertices of a regular hexagon side a.

(i)



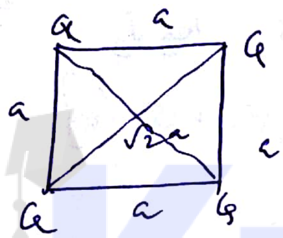
$$W_E = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}} = \frac{Q^2}{4\pi\varepsilon_0 a}$$

(ii)



$$W_E = \frac{Q^2}{4\pi\varepsilon_0 a} \times 3$$

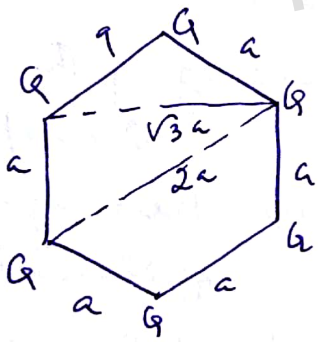
(iii)



$$W_E = \frac{Q^2}{4\pi\varepsilon_0 a} \times 4 + \frac{Q^2}{4\pi\varepsilon_0 \sqrt{2}a} \times 2$$

$$= \frac{4Q^2}{4\pi\varepsilon_0 a} + \frac{2Q^2}{4\pi\varepsilon_0 \sqrt{2} a}$$

(iv)

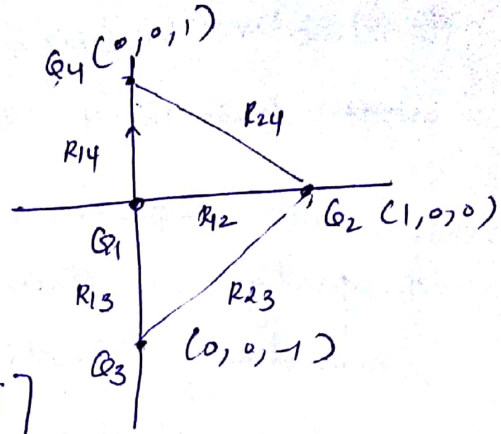


$$W_E = \frac{Q^2 \times 6}{4\pi\varepsilon_0 a} + \frac{Q^2}{4\pi\varepsilon_0 \sqrt{3}a} \times 6 + \frac{Q^2 \times 3}{4\pi\varepsilon_0 2a}$$

n sides mean

$\frac{n(n-1)}{2}$ terms.

Point charges $Q_1 = 1nC$, $Q_2 = -2nC$, $Q_3 = 3nC$ and $Q_4 = -4nC$ are placed one by one in the same order at $(0,0,0)$ $(1,0,0)$, $(0,0,-1)$ and $(0,0,1)$ resp: calculate the energy in the system when all charges are placed.



$$W_E = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1Q_2}{R_{12}} + \frac{Q_2Q_3}{R_{23}} + \frac{Q_3Q_1}{R_{13}} + \frac{Q_1Q_4}{R_{14}} + \frac{Q_2Q_4}{R_{24}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1\times10^{-9}\times(-2\times10^{-9})}{1} + \frac{(-2\times10^{-9})\times 3\times10^{-9}}{\sqrt{2}} + \frac{3\times10^{-9}\times 1\times10^{-9}}{1} + \frac{1\times10^{-9}\times(-4\times10^{-9})}{1} + \frac{-2\times10^{-9}\times(-4\times10^{-9})}{\sqrt{2}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-2\times10^{-18} + 3\times10^{-18} - 4\times10^{-18} - \frac{6\times10^{-18}}{\sqrt{2}} + \frac{8\times10^{-18}}{\sqrt{2}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-3\times10^{-18} + \frac{2\times10^{-18}}{\sqrt{2}}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{-4.24\times10^{-18} + 2\times10^{-18}}{\sqrt{2}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{(-2.24\times10^{-18})}{\sqrt{2}}$$

$$= 9\times10^{9}\left[\frac{(-2.24\times10^{-18})}{\sqrt{2}} + \frac{-12\times10^{-18}}{2}\right]$$

$$= -68.25\,J$$

Boundary Conditions

When 2 kinds of media meet at a common boundary the electromagnetic properties change abruptly across the boundary of separation. Dielectric - vacuum, conductor - vacuum, dielectric conductor etc are some examples of boundaries.

An expression that relates the components of a property on either sides of a boundary is called a boundary condition.

The electromagnetic property like E, D, B and H varies in a small region around the boundary. We consider the idealised situation in which the property undergoes a sudden change.

When an electric field passes from one medium to another it is important to study the conditions at the boundary between the 2 media. The conditions existing at the boundary of the 2 media when field passes from one medium to other are called boundary conditions.

(i) Boundary between conductor & free space (dielectric)

(ii) Boundary between 2 dielectrics with diff: properties

For studying boundary conditions we need the relations

$$\oint E \cdot dl = 0 \qquad \oint D \cdot ds = 0$$

E is decomposed into tangential & normal component

ie $\bar{E} = \bar{E}_{tan} + \bar{E}_N$

|||ly D as well

Boundary Conditions between Conductor & dielectric

Consider a boundary between conductor & free space. The conductor is ideal having infinite conductivity. eg copper, silver etc having conductivity of the order of 10^5 S/m and can be treated ideal. For ideal conductors

(i) field intensity inside a conductor is zero and flux density ~~on the~~ surface inside the conductor is zero.

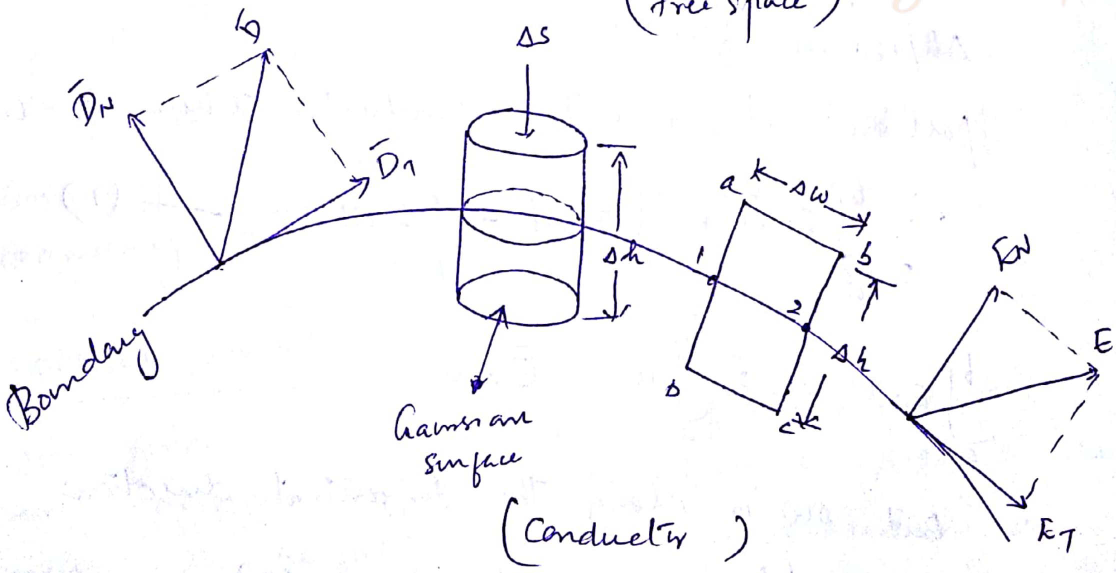
(ii) No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density.

(iii) Charge density within the conductor is zero

Then E, D and ρ_v within the conductor are zero where ρ_s is the surface charge density on the surface of the conductor.

To determine the boundary conditions consider the closed path and Gaussian surface.

(free space)



(Conductor)

Consider the boundary as shown in fig.

E at the boundary

Let $\bar{E}$ be the electric field intensity. This $\bar{E}$ can be resolved into 2 components

(i) Tangential to the surface E_T

(ii) Normal to the surface E_N

It is known that

$$\oint E \cdot dl = 0$$

Consider a rectangular path (closed) abcda. It is traced in clockwise direction as a b c d a and $\oint E \cdot dl$ can be divided into

$$\oint E \cdot dl = \int_a^b E \cdot dl + \int_b^c E \cdot dl + \int_c^d E \cdot dl + \int_d^a E \cdot dl = 0$$

The rectangle is an elementary rectangle with height Δh and width Δw. The rectangle is placed in such a way that ½ of it is in conductor and remaining half is in free space. Thus $\Delta h/2$ is in conductor and $\Delta h/2$ is in free space.

Portion cd is in the conductor where $E = 0$

$$\therefore \int_a^b \bar{E} \cdot d\bar{l} + \int_b^c \bar{E} \cdot dl + \int_d^a E \cdot dl = 0 \quad \text{——(1)}$$

$$\int_a^b E \cdot dl = \bar{E} \int_a^b dl = \bar{E}\Delta w$$

But Δw is along the tangential direction

$$\therefore \int_a^b E \cdot dl = E_T(\Delta w) \quad \text{——(2)}$$

Now bc is $\parallel^{el}$ to the normal component so

$$\oint_b E \cdot dl = E \cdot \frac{\Delta h}{2} = E_N \left(\frac{\Delta h}{2}\right) \quad \text{——(3)}$$

||ly for path da

$$\int_d^a E \cdot dl = -E_N \left(\frac{\Delta h}{2}\right) \quad \text{——(4)}$$

sub 2, 3, 4 in (1)

$$\bar{E}_T \Delta w + E_N \left(\frac{\Delta h}{2}\right) - E_N \left(\frac{\Delta h}{2}\right) = 0$$

$$\therefore \bar{E}_T (\Delta w) = 0$$

$$\boxed{\therefore \bar{E}_T = 0}$$

Thus the tangential component of $\bar{E}$ is zero at the boundary between conductor & free space.

ie $\bar{E}$ at the boundary between conductor & free space is always in the direction ⊥r to the boundary.

Now $D = \varepsilon_0 \bar{E}$

$$\boxed{\therefore D_T = \varepsilon_0 E_T = 0}$$

$\therefore$ Tangential component of $\bar{D}$ is zero at the boundary and hence $\bar{D}$ is also normal at the boundary between conductor & free space.

D_N at the boundary

To find normal component of $\bar{D}$ select a closed Gaussian surface in the form of right circular cylinder. Its height is Δh and is placed in such a way that $\Delta h/2$ is in conductor and remaining $\Delta h/2$ is in free space. Its axis is in the normal direction to the surface.

According to Gauss law $\oint_S D \cdot ds = Q$

Surface charge integral must be evaluated over 3 surfaces: (i) top (ii) bottom and (iii) lateral.

Let the area of top and bottom is same equal to ΔS

$$\therefore \int_{top} \bar{D} \cdot ds + \int_{bottom} \bar{D} \cdot ds + \int_{lateral} \bar{D} \cdot ds = Q$$

At the bottom surface $D = 0$

Top surface is in the free space and we are interested in boundary condition hence top surface can be shifted at the boundary with $\Delta h \to 0$

$$\therefore \int_{top} \bar{D} \cdot ds + \int_{lateral} \bar{D} \cdot ds = Q$$

lateral surface area is $2\pi r \Delta h$ But as $\Delta h \to 0$ this area reduces to zero and corresponding integral is zero.

The only component of $\bar{D}$ present is the normal component having magnitude D_N

$$\therefore \int_{top} \bar{D} \cdot d\bar{s} = D_N \int_{top} ds = D_N \cdot \Delta S$$

From Gauss law

$$\therefore D_N \Delta S = Q$$

$$D_N \Delta S = \rho_s \Delta s$$

$$\boxed{\therefore D_N = \rho_s}$$

Thus the flux leaves the surface normally and the normal component of flux density is equal to the surface charge density.

$D_N = \varepsilon_0 E_N = \rho_s$

$$\boxed{E_N = \frac{\rho_s}{\varepsilon_0}}$$

As the tangential component of E is zero surface of the conductor is an equipotential surface.

Boundary conditions are

$$\boxed{\begin{aligned} &E_T = D_T = 0 \\ &D_N = \rho_s \\ &E_N = \frac{\rho_s}{\varepsilon} = \frac{\rho_s}{\varepsilon_0 \varepsilon_r} \end{aligned}}$$

1) A potential field is given by $V = 100 e^{-5x} \sin 3y \cos 4z$ V. Let point $P(0.1, \pi/12, \pi/24)$ be located at a conductor free space boundary. At point P find the magnitude of (i) V, $\bar{E}$, E_t, $\bar{D}$, E_N, D_N, ρ_s

At P $x = 0.1$, $y = \pi/12$, $z = \pi/24$

(radian mode)

(i) $V = 100 e^{-0.5} \sin \frac{3\pi}{12} \cos \frac{4\pi}{24} = \underline{\underline{37.14 \text{ V}}}$

(ii) $E = -\nabla V = -\left(\frac{\partial V_x}{\partial x} \hat{i} + \frac{\partial V_y}{\partial y} \hat{j} + \frac{\partial V_z}{\partial z} \hat{k}\right)$

$= -100\left(-5 e^{-5x} \sin 3y \cos 4z \, \hat{i} + e^{-5x} (3) \cos 3y \, (\cos 4z) \hat{j} + e^{-5x} \sin 3y \cdot 4(-\sin 4z) \hat{k}\right]$

$= -100\left[-1.857\hat{i} + 1.114\hat{j} - 0.857\hat{k}\right)$

$= \cancel{185.7\hat{j} - 111.4}\; 185.7\hat{i} - 111.4\hat{j} + 85.77\hat{k}$

$|\bar{E}| = \underline{232.92 \text{ V/m}}$

(iii) $E_t = 0$ V/m

(iv) $E_N = |\bar{E}| = 232.92$ V/m

(v) $\bar{D} = \varepsilon_0 E = 8.854\times10^{-12} \times (185.7\hat{i} - 111.4\hat{j} + 85.77\hat{k})$

$= 1.588\hat{i} - 0.952\hat{j} + 0.733\hat{k}$ nC/m^2

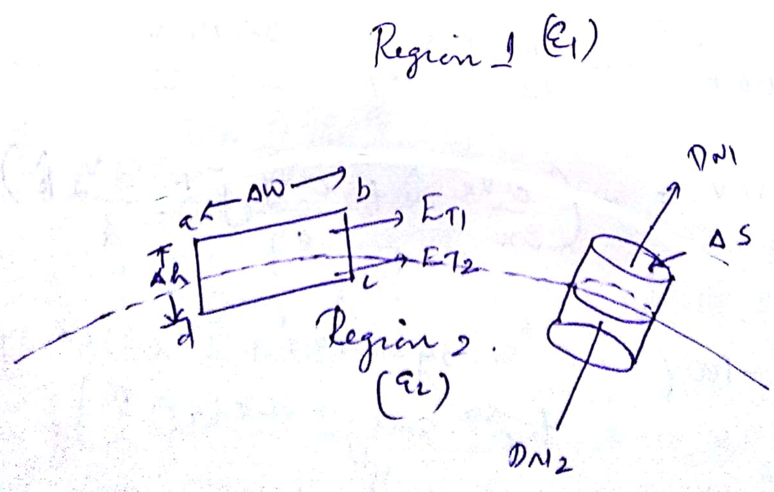
$|\bar{D}| = 1.992$ nC/m^2

$D_N = |\bar{D}| = 1.992$ nC/m^2

$D_N = \rho_s = 1.992$ nC/m^2

Boundary conditions between 2 perfect Dielectrics

Let us consider the boundary between 2 perfect dielectrics. One dielectric has permittivity ε_1 while the other has permittivity ε_2.



$\bar{E}$ and $\bar{D}$ are to be obtained again by resolving each into 2 components, tangential to the boundary and normal to the surface.

Consider a closed path abcda rectangular in shape having elementary height Δh and width Δw. It is placed in such a way that $\Delta h/2$ is in dielectric 1 while the remaining is dielectric 2.

$$\oint E \cdot dl = 0$$

$$\therefore \int_a^b E \cdot dl + \int_b^c E \cdot dl + \int_c^D E \cdot dl + \int_d^a E \cdot dl = 0 \quad \text{---(1)}$$

$$E_1 = \bar{E}_{1t} + \bar{E}_{1N} \qquad |E_{1t}| = E_{t1} \qquad |E_{1N}| = E_{N1}$$

$$E_2 = E_{2t} + E_{2N} \qquad |E_{2t}| = E_{t2} \qquad |E_{2N}| = E_{N2}$$

Rectangle abcd is reduced at the surface to analyse boundary conditions, $\Delta h \to 0$. As $\Delta h \to 0$ $\therefore \int_b^c$ and $\int_d^a$ becomes zero

$$\therefore \int_a^b \bar{E} \cdot d\bar{l} + \int_c^d \bar{E} \cdot d\bar{l} = 0 \quad \text{---(2)}$$

a-b is in dielectric 1 hence the corresponding component of E is $\bar{E}_{t1}$ on a-b direction is tangential to the surface.

$$\therefore \int_a^b E \cdot d\bar{l} = E_{t1} \int_a^b dl = E_{t1} \cdot \Delta w \quad \text{---(3)}$$

$$\text{||}^{ly} \quad \int_c^d E \cdot dl = E_{t2} \int_c^d dl = -E_{t2} \, \Delta w \quad \text{---(4)}$$

sub (3) & (4) in (2)

$$E_{t1} \cdot \Delta w - E_{t2} \Delta w = 0 \quad ; \quad \boxed{E_{t1} = E_{t2}}$$

Thus the tangential components of $\bar{E}$ at the boundary in both the dielectrics remain same ie E is continuous across the boundary

$$D = \varepsilon E$$

$$\therefore D_{t1} = \varepsilon_1 E_{t1}$$

$$D_{t2} = \varepsilon_2 E_{t2}$$

$$\boxed{\frac{D_{t1}}{D_{t2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$

Thus tangential components of $\bar{D}$ undergoes some change across the interface hence tangential $\bar{D}$ is said to be discontinuous across the boundary.

To find the normal Components

Consider the Gaussian surface in the form of right circular cylinder placed in such a way that half of it lies in dielectric 1 and remaining half in dielectric 2. The height $\Delta h \to 0$ hence flux leaving from its lateral surface is zero. Surface area of top & bottom is ΔS

$$\therefore \oint \bar{D} \cdot ds = Q$$

$$\therefore \int_{top} \bar{D} \cdot ds + \int_{bottom} \bar{D} \cdot ds + \int_{lateral} \bar{D} \cdot ds = Q$$

But $\int_{latu} D \cdot ds = 0$ as $\Delta h \to 0$

$$\therefore \int_{top} \bar{D} \cdot ds + \int_{bottom} \bar{D} \cdot ds = Q$$

The flux leaving normal to the boundary is normal to the top and bottom surfaces

$\therefore |\bar{D}| = D_{N1}$ for dielectric 1

D_{N2} for dielectric 2

$$\therefore \int_{top} \bar{D}\cdot d\bar{s} = D_{N1} \int_{top} d\bar{s} = D_{N1} \cdot \Delta S$$

$$\text{IIIly} \int_{bottom} \bar{D}\cdot ds = -D_{N2} \int_{bottom} ds = -D_{N2} \cdot \Delta S$$

$$\therefore D_{N1} \Delta S - D_{N2} \Delta S = Q$$

$$(D_{N1} - D_{N2}) \Delta S = \rho_s \cdot \Delta S$$

$$D_{N1} - D_{N2} = \rho_s$$

There is no free charge available in perfect dielectric hence no free charge can exist on the surface

$\therefore \rho_s = 0$

$$\therefore D_{N1} - D_{N2} = 0$$

$$\boxed{D_{N1} = D_{N2}}$$

$\therefore \bar{D}$ is continuous at the boundary between 2 perfect dielectrics

$$D_{N1} = \epsilon_1 E_{N1}$$

$$D_{N2} = \epsilon_2 E_{N2}$$

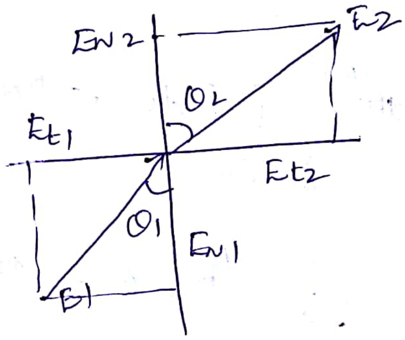
$$\frac{D_{N1}}{D_{N2}} = \frac{\epsilon_1}{\epsilon_2} \frac{E_{N1}}{E_{N2}}$$

∴ $D_{N1} = D_{N2}$

$$\frac{\epsilon_1}{\epsilon_2}\frac{E_{N1}}{E_{N2}} = 1$$

$$\boxed{\frac{E_{N1}}{E_{N2}} = \frac{\epsilon_2}{\epsilon_1} = \frac{\epsilon_{r2}}{\epsilon_{r1}}}$$

∴ Normal component of $\bar{E}$ are inversely ∝ to relative permittivities of the 2 media.



We have $E_{t1} = E_{t2}$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

$D_{N1} = D_{N2}$

ie $\epsilon_1 E_{N1} = \epsilon_2 E_{N2}$

$$\epsilon_1 E_1 \cos\theta_1 = \epsilon_2 E_2 \cos\theta_2$$

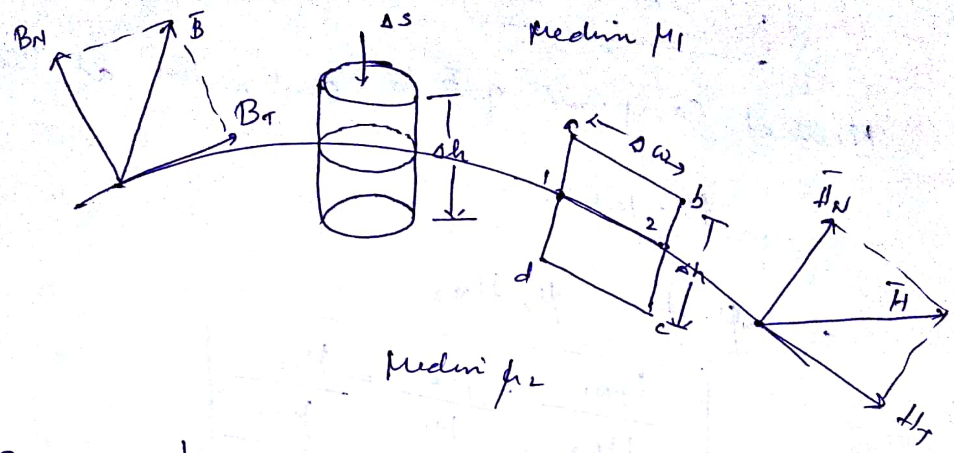
From these we have

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$ (Law of refraction).

Magnetic Boundary Conditions

The conditions of the magnetic field existing at the boundary of the 2 media when the magnetic field passes from one medium to other are called boundary conditions for magnetic fields or magnetic boundary conditions. The boundary between 2 different magnetic materials is considered. Both $\bar{B}$ and $\bar{H}$ are resolved into tangential and normal components.

...nsider a Boundary between 2 isotropic homogeneous medium with diff: permeabilities μ_1 and μ_2. A closed path and Gaussian surface is considered



Normal Component

To find the normal component of $\bar{B}$ select a closed Gaussian surface in the form of a right circular cylinder. Let height be Δh and be placed in such a way that $\Delta h/2$ is in medium 1 and remaining $\Delta h/2$ in medium 2. axis of the cylinder is in the normal direction to the surface.

According to Gauss law for magnetic field

$$\oint_S B \cdot ds = 0$$

ie $$\oint_{top} \bar{B} \cdot d\bar{s} + \oint_{bottom} B \cdot d\bar{s} + \oint_{lateral} \bar{B} \cdot ds = 0 \quad \text{--- (1)}$$

to get boundary condition $\Delta h \to 0$ so only top and bottom surfaces contribute in the surface integral.

$$\oint_{top} \bar{B} \cdot ds = B_{N1} \oint_{top} ds = B_{N1} \cdot \Delta S \qquad \oint_{lateral} \bar{B} \cdot d\bar{s} = 0$$

$$\oint_{bottom} B \cdot ds = -B_{N2} \oint_{bottom} ds = B_{N2} \cdot \Delta S$$

So ① becomes

(-ve sign on component of 2 is entering as is leaving)

$$B_{N1}\,\Delta S - B_{N2}\,\Delta S = 0$$

$$\boxed{\therefore\ B_{N1} = B_{N2}}$$

Normal component of $\bar{B}$ is continuous at the boundary.

$$B = \mu H$$

$$\mu_1 H_{N1} = \mu_2 H_{N2}$$

$$\boxed{\frac{H_{N1}}{H_{N2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r2}}{\mu_{r1}}}$$

$\bar{H}$ is not continuous at boundary.

Tangential Components

According to ampere's circuital law

$$\oint H\cdot dl = I$$

Consider the rectangular closed path abcda

$$\oint H\cdot dl = \int_a^b H\cdot dl + \int_b^1 H\cdot dl + \int_1^c H\cdot dl + \int_c^d H\cdot dl + \int_d^2 H\cdot dl + \int_2^a H\cdot dl = I$$

$$H_{t1}\,\Delta w + H_{N1}(\Delta h/2) + H_{N2}\,\Delta h/2 - H_{t2}\,\Delta w - H_{N2}(\Delta h/2) - H_{N1}(\Delta h/2)$$
$$= k\cdot dw \qquad (k\ \text{surface current})$$

As $\Delta h \to 0$

$$k\cdot dw = H_{t1}\,\Delta w - H_{t2}\,\Delta w$$

$$\therefore\ H_{t1} - H_{t2} = k$$

In vector form $\boxed{H_{t1} - H_{t2} = \bar{a}_{N12} \times \bar{k}}$

$$\therefore \frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = k$$

Consider a special case that the boundary is free of current. ie media are not conductors So $k=0$

$$\therefore H_{t1} - H_{t2} = 0$$

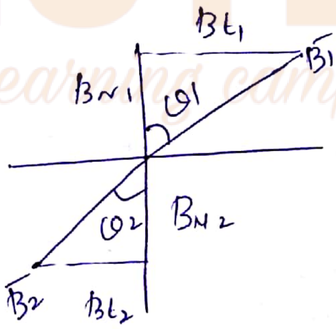
$$H_{t1} = H_{t2}$$

$$\therefore \frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = 0$$

$$\frac{B_{t1}}{\mu_1} = \frac{B_{t2}}{\mu_2}$$

$$\boxed{\frac{B_{t1}}{B_{t2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}}$$

Let the field makes angles θ_1 and θ_2 with the normal to the interface



In medium 1 $\tan\theta_1 = \frac{B_{t1}}{B_{N1}}$

lly $\tan\theta_2 = \frac{B_{t2}}{B_{N1}}$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{B_{t1}}{B_{t2}} \cdot \frac{B_{N2}}{B_{N1}}$$

$$\boxed{\text{ie } \frac{\tan\theta_1}{\tan\theta_2} = \frac{B_{t1}}{B_{t2}} = \frac{\mu_{r1}}{\mu_{r2}}}$$

$[\because B_{N1} = B_{N2}]$

Conduction current and displacement currents

Conduction current : Present in conductors & semiconductors and carried by drift motion of $\bar{e}$ & holes in response to an applied field

Current I flowing through a conductor whose resistance is R when potential V is applied across the conductor

$$I_c = V/R \qquad R = \frac{\rho l}{A}$$

$$R = \frac{l}{\sigma A} \qquad \sigma = 1/\rho$$

If E is the electric field existing in the conductor, the potential $V = El$

$$I_c = V/R = \frac{El}{R} = \frac{El\,\sigma A}{l} = \underline{E\sigma A}$$

Conduction current density

$$J_c = \frac{I_c}{A} = \frac{E\sigma A}{A} = \sigma E$$

$\boxed{\text{i.e. } J = \sigma E}$ point form of Ohm's law. It states that the field strength within a conductor is $\propto$ to the current density.

$J \propto E$

$J = \sigma E \qquad \sigma = \frac{1}{\rho} \qquad \sigma$ - conductivity reciprocal of resistivity

Displacement current : Result of migration of +ve & -ve ion as well known as time varying field phenomenon

Displacement current I_D is flowing through a capacitor when ac voltage is applied across the capacitor) that allow current to flow between plates of a

Dipole orientation

$$I_D = \frac{dQ}{dt}$$

$$= C\frac{dv}{dt} \quad (Q = CV)$$

For parallel plate capacitor

$$C = \frac{\varepsilon A}{d}$$

$$\therefore I_D = \frac{\varepsilon A}{d} \cdot \frac{dv}{dt}$$

$$= \varepsilon A \cdot \frac{dE}{dt} \quad (\because V = Ed)$$

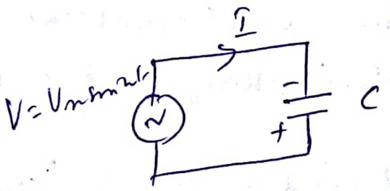
$$D = \varepsilon E$$

$$= \varepsilon A \cdot \frac{d\, D/\varepsilon}{dt}$$

$$I_D = A \cdot \frac{dD}{dt}$$

Displacement current density $J_D = \frac{\partial D}{\partial t}$.

4) Show that the displacement current through the parallel plate capacitor is equal to the conduction current I flowing in the external circuit.



$$I_D = \int_S J_D \cdot ds$$

$$= \int_S \frac{\partial D}{\partial t} \cdot ds$$

$$= \frac{\partial D}{\partial t} \int_S ds = \frac{\partial D}{\partial t} \cdot A$$

$$E = V/d \qquad D = \varepsilon E$$

$$D = \frac{\varepsilon V}{d}$$

differentiate w.r.t 't'

$$\frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{\partial v}{\partial t}$$

$$V = V_m \sin\omega t$$

$$\frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{\partial (V_m \sin\omega t)}{\partial t}$$

$$\frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\omega V_m \cos\omega t$$

$$\therefore I_D = A\frac{\varepsilon}{d}\omega V_m \cos\omega t$$

$$I_C = C\frac{dv}{dt} \quad ; \quad C = \frac{\varepsilon A}{d}$$

$$\frac{dv}{dt} = \omega V_m \cos\omega t$$

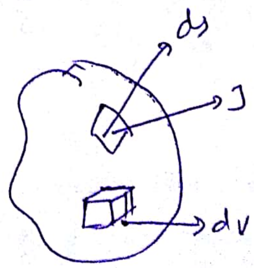
$$I_C = \frac{\varepsilon A}{d}\omega V_m \cos\omega t$$

$$\therefore I_D = I_C$$

Equation of Continuity

It is the mathematical statement of the law of conservation of electric charges for time varying systems.

$$I = \oint_S J \cdot ds$$ (Current through diff: area ds at point p on a closed surface S)



Let ρ_v be the volume charge density

$$\therefore Q = \int_V \rho_v \, dv$$

Law of Conservation of charge demands that the total charge that flows out through the surface in unit time is equal to the time rate of decrease of charge contained in the volume

$$\oint_s J \cdot ds = \frac{-\partial}{\partial t} \int_v \rho_v \, dv$$

using divergence theorem

$$\int_v \nabla \cdot J \, dv = \frac{-\partial}{\partial t} \int_v \rho_v \, dv$$

$$\int_v \nabla \cdot J \, dv = \int_v \frac{-\partial \rho_v}{\partial t} dv$$

ie $\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}$

$\nabla \cdot J + \frac{\partial \rho_v}{\partial t} = 0$ Equation of continuity

First term gives the net out flux of charges / unit volume in unit time and second term gives the rate of increase of charge / unit volume. The sum of these must be zero due to conservation of charges.

Maxwell's Equation

A static electric field E can exist without a magnetic field H. Similarly a conductor with a constant current I has a magnetic field H in the absence of an electric field. But in case of time varying fields E and H are interdependent.

Maxwell's equations are derived from Ampere's circuit law, Faraday's law, Gauss's law for electric field and Gauss's law for magnetic field. Four expressions can be written in

(i) Point or differential form

(ii) Integral form.

(1) According to Ampere's circuital law, the line integral of magnetic field intensity H around a closed path is equal to the current enclosed by the path.

ie $\oint H \cdot dl = I = \int_S J \cdot ds$

current is comprised of conduction current and displacement current

ie $I = I_c + I_D = \int_S J_c \cdot ds + \int_S J_D \cdot ds$

$I_c = \int_S J \cdot ds \qquad J = \sigma E$

$I_D = \int_S J_D \cdot ds \qquad J_D = \frac{\partial D}{\partial t}$

$$\boxed{\therefore \oint H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t} \right) \cdot ds}$$

Maxwell's equation from Ampere's law in integral form

Applying Stokes theorem

$$\int_S (\nabla \times H) \cdot ds = \int_S \left(J + \frac{\partial D}{\partial t} \right) \cdot ds$$

$$\boxed{\nabla \times H = J + \frac{\partial D}{\partial t}}$$ differential form.

(2) According to Faraday's law the emf induced into a circuit is equal to the rate of decrease of flux linking the circuit.

$$e = -N\frac{d\phi}{dt}$$

If $N = 1$

$$e = -\frac{d\phi}{dt}$$

$$\therefore e = \oint E \cdot dl$$

$$\phi = \int_S \bar{B} \cdot d\bar{s}$$

$$\therefore \oint E \cdot dl = -\frac{d}{dt}\int_S B \cdot ds$$

$$\boxed{\oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot ds}$$

Maxwell's equation in integral form from Faraday's law.

Applying Stoke's theorem

$$\int_S (\nabla \times E) \cdot ds = -\int_S \frac{\partial B}{\partial t} \cdot ds$$

$$\boxed{\text{ie } \nabla \times E = -\frac{\partial B}{\partial t}}$$ — Differential form

(3) From Gauss's law the electric flux through any closed surface is equal to the charge enclosed by the surface

$$\psi = Q$$

$$\boxed{\int_S D \cdot ds = \int_V \rho_v \, dv}$$

Maxwell's equation in integral form from Gauss's law for electric field

Applying Divergence theorem

$$\int_V \nabla \cdot D \, dv = \int_V \rho_v \, dv$$

$$\boxed{\nabla \cdot D = \rho_v}$$ Differential form.

(A) From Gauss's law for magnetic circuit which states that The magnetic flux through any closed surface is equal to zero since no isolated magnetic monopole exists

$$\boxed{\int_S B \cdot ds = 0}$$ Maxwells equation in integral form from magnetic Gauss's law

Applying divergence theorem

$$\int_V (\nabla \cdot B) \, dv = 0$$

$$\boxed{\text{ie } \nabla \cdot B = 0}$$ differential form.

Integral form	Differential form	Law
$\oint H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t}\right) \cdot ds$	$\nabla \times H = J + \frac{\partial D}{\partial t}$	Ampere's law
$\oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot ds$	$\nabla \times E = \frac{-\partial B}{\partial t}$	Faraday's law
$\int_S D \cdot ds = \int_V \rho_v \, dv$	$\nabla \cdot D = \rho_v$	Gauss's law
$\int_S B \cdot ds = 0$	$\nabla \cdot B = 0$	Magnetic Gauss's law

In case of free space

$$\nabla \times H = J$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot D = 0$$

$$\nabla \cdot B = 0$$

1) Find the amplitude of the displacement current density
(i) in the air near car antenna where the field strength of FM signal is

$$E = 80 \cos(6.277 \times 10^8 t - 2.092 y)\, \hat{k} \; V/m$$

(ii) Inside a capacitor where $\epsilon_r = 600$ and

$$\bar{D} = 3 \times 10^{-6} \sin(6 \times 10^6 t - 0.3464 x)\, \hat{k} \; C/m^2$$

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon_0 \epsilon_r E)$$

(i) For air $\epsilon_r = 1$

$$\therefore J_D = \epsilon_0 \frac{\partial E}{\partial t}$$

$$= 8.854 \times 10^{-12} \frac{\partial}{\partial t}\left[80 \cos(6.277 \times 10^8 t - 2.092 y)\hat{k}\right]$$

$$= 8.854 \times 10^{-12}\, 80\, (-6.277 \times 10^8) \sin(6.277 \times 10^8 t - 2.092 y)\hat{k}$$

$$= -0.4446 \sin(6.277 \times 10^8 t - 2.092 y)\hat{k}$$

$\therefore$ amplitude is $J_D = \underline{\underline{0.4446 \; A/m^2}}$

(ii) For capacitor $\epsilon_r = 600$

$$J_D = \frac{\partial}{\partial t}\left(3 \times 10^{-6} \sin(6 \times 10^6 t - 0.3464 x)\hat{k}\right)$$

$$= (3 \times 10^{-6})(6 \times 10^6) \cos(6 \times 10^6 t - 0.3464 x)\hat{k}$$

$= 18 \cos(6\times10^{6} t - 0.3464x)\, \hat{k}$

$\therefore J_D = 18 \text{ A/m}^2$