AWH ENGINEERING COLLEGE

Additional Answer Sheet
(Two Writing Pages)

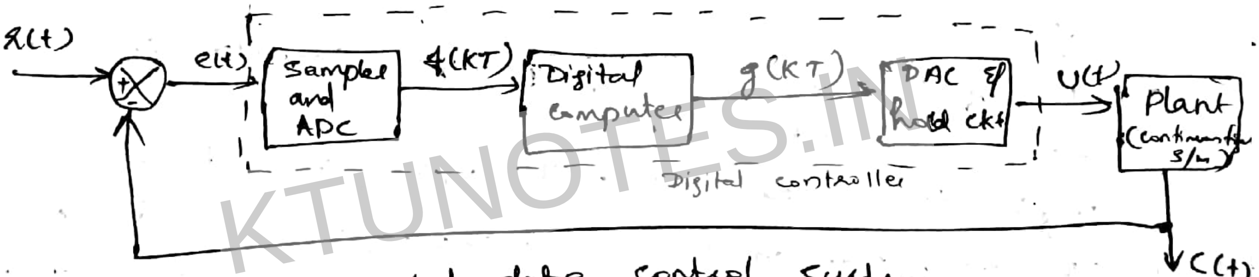
Module IV

1

SAMPLED DATA CONTROL SYSTEM

When the signal or information at any or some points in a system is in the form of discrete pulses, then the system is called discrete data system. It is also called sampled data system.

The controllers are provided in control systems to modify the error signal for better control action. The digital controllers are simple, versatile, programmable, fast acting and less costlier than analog controllers.



Sampled data control system

e(t) = error signal
u(t) = control signal } Analog in nature.

The sampler converts the continuous time-error signal in to a sequence of pulses and ADC produces a binary code for each sample. Digital computer processes the binary codes and produces another stream of binary codes as output. DAC and hold ckt converts the output binary codes to continuous time signal (Analog signal) called control signal. This output control signal is used to drive the plant.

ADVANTAGES OF DIGITAL CONTROLLERS

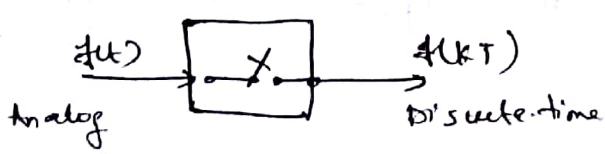
1. They perform large and complex computation with any desired degree of accuracy at very high speed. In analog controllers, the cost of controllers increases rapidly with increase in complexity of computation and desired accuracy.
2. Digital controllers are easily programmable and so they are more versatile.
3. Digital controllers have better resolution.

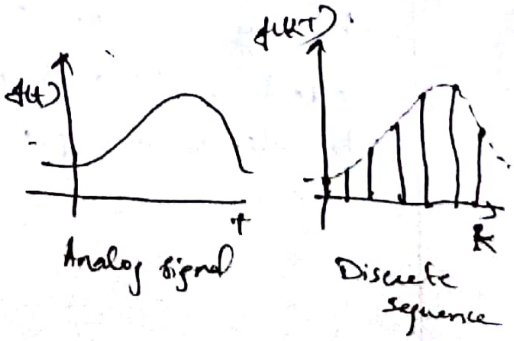
ADVANTAGES OF SAMPLED DATA CONTROL SYSTEM

1. The sampled data systems are highly accurate, fast and flexible.
2. Use of time sharing concept of digital computers results in economical cost and space.
3. Digital transducers used in the system have better resolution.
4. The digital components used in the systems are less affected by noise, nonlinearities and transmission errors of noisy channel.
5. The sampled data systems require low power instruments which can be built to have high sensitivity.
6. Digital coded signals can be stored, transmitted, retransmitted, detected, analysed or processed as desired.
7. The system performance can be modified by compensation technique.

Sampling Process.

It is the conversion of a continuous time signal into discrete time signal by taking samples of continuous signal at discrete instants.





The sampling frequency $F_s = 1/T$ must be selected large enough such that the sampling process will not result in any loss of spectral information.

Sampling theorem

A band limited continuous time signal with highest frequency (f_m) hertz, can be uniquely recovered from its samples provided that the sampling rate F_s is greater than or equal to $2f_m$ samples per second.

Discrete time signal

A discrete sequence or discrete time signal, $f(k)$ is a function of an independent variable, k, which is an integer.

A discrete-time signal is defined for every integer value of k in the range $-\infty < k < \infty$.

Since a digital signal is represented by a set of numbers it is also called a sequence.

Z transform :

$f(k)$ = Discrete sequence

$F(z) = Z\{f(k)\}$ = Z transform of $f(k)$

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \quad \text{(2 sided)}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k} \quad \text{(One sided)}$$

Region of convergence

ROC of $F(z)$ is the set of all values of Z, for which $F(z)$ attains a finite value.

The ROC of a finite duration signal is the entire Z-plane, except possibly the point $z = 0$ and/or $z = \infty$

$Z = re^{j\omega}$

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{\infty} f(k)(re^{j\omega})^{-k}$$

$$|F(z)| = \sum_{k=-\infty}^{\infty} |f(k) r^{-k}|$$

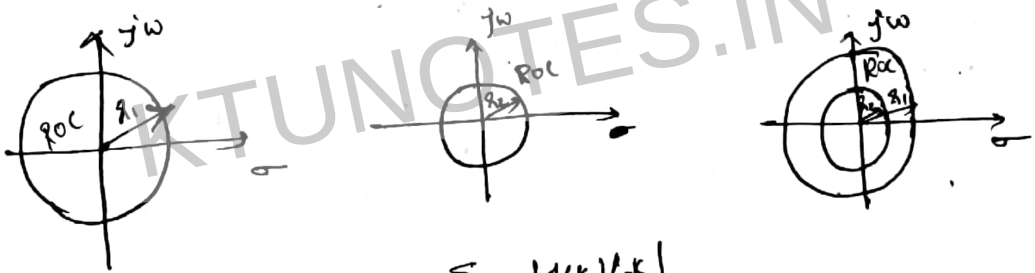
$$= \sum_{k=-\infty}^{-1} |f(k) r^{-k}| + \sum_{k=0}^{\infty} |f(k) r^{-k}|$$

$$= \sum_{k=1}^{\infty} |f(-k) r^{k}| + \sum_{k=0}^{\infty} \left| \frac{f(k)}{r^k} \right|$$

Roc of 1st term $\Longrightarrow$ All points in a circle of radius, r_1

Roc for 2nd term $\Longrightarrow$ All points in a circle of radius r_2

ie $r_2 < r < r_1$



$\sum_{k=1}^{\infty} |f(-k) r^k|$ $\quad\quad$ $\sum_{k=0} |f(k)/r^k|$

TABLE-3.1 : CHARACTERISTIC FAMILIES OF SIGNALS WITH THEIR CORRESPONDING ROC

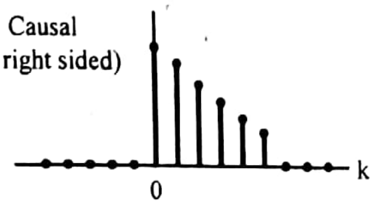
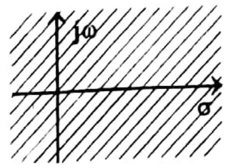
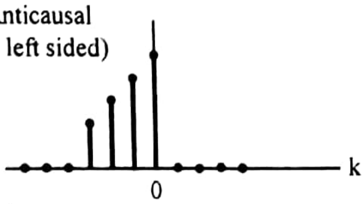
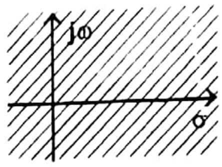
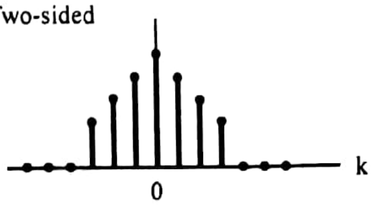
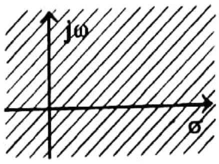
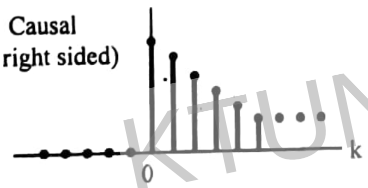
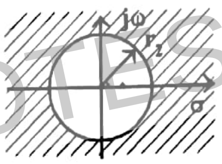
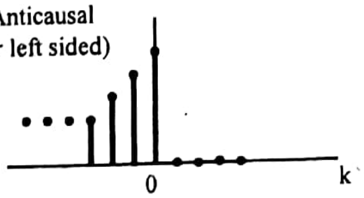
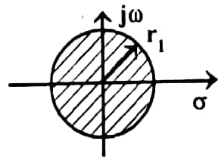
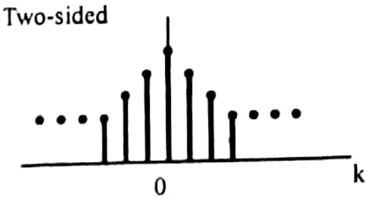
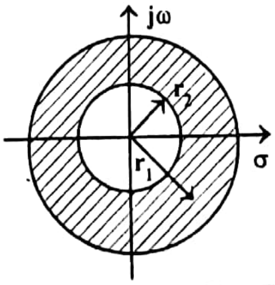
SIGNAL	ROC
Finite-Duration Signals	
Causal (or right sided) 	 Entire z-plane except z= 0
Anticausal (or left sided) 	 Entire z-plane except z= ∞
Two-sided 	 Entire z-plane except z = 0 and z = ∞
Infinite-Duration Signals	
Causal (or right sided) 	 $\lvert z\rvert > r_2$
Anticausal (or left sided) 	 $\lvert z\rvert < r_1$
Two-sided 	 $r_2 < \lvert z\rvert < r_1$

TABLE-3.2 : PROPERTIES OF ONE-SIDED Z-TRANSFORM

Notations $F(z) = \mathcal{Z}\{f(k)\}$; $F_1(z) = \mathcal{Z}\{f_1(k)\}$; $F_2(z) = \mathcal{Z}\{f_2(k)\}$

Property	Discrete sequence	z-transform
Linearity	$a_1 f_1(k) + a_2 f_2(k)$	$a_1 F_1(z) + a_2 F_2(z)$
Shifting, $m \geq 0$	$f(k+m)$ $f(k-m)$	$z^m F(z) - \sum_{i=0}^{m-1} f(i) z^{m-i}$ $z^{-m} F(z)$
Multiplication by k^m (or differentiation in z-domain)	$k^m f(k)$	$\left(-z\frac{d}{dz}\right)^m F(z)$
Scaling in z-domain (or multiplication by a^k)	$a^k f(k)$	$F(a^{-1}z)$
Time reversal	$f(-k)$	$F(z^{-1})$
Conjugation	$f^*(k)$	$F^*(z^*)$
Convolution	$\sum_{m=0}^{k} h(k-m)\, r(m)$	$H(z)\, R(z)$
Initial value	$f(0) = \underset{z\to\infty}{Lt}\ F(z)$	$\underset{z\to\infty}{Lt}\ F(z)$
Final value	$f(\infty) = \underset{z\to1}{Lt}\,(1-z^{-1})\,F(z)$ $= \underset{z\to1}{Lt}\,(z-1)\,F(z)$ if $F(z)$ is analytic for $\|z\| > 1$	

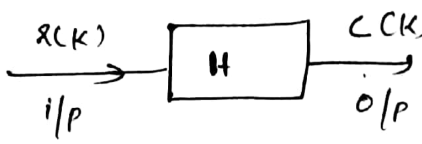
TABLE-3.3 : SOME COMMON ONE SIDED Z-TRANSFORM PAIRS

Note : Two sided sequence can be converted to one sided sequence by multiplying by u(k).

$f(t)\ ;\ t \geq 0$	$f(k)$ or $f(kT)\ ;\ k \geq 0$	$F(z)$
	$\delta(k)$	1
	$u(k)$ or 1	$z/(z-1)$
	a^k	$z/(z-a)$
	$k\,a^k$	$\dfrac{az}{(z-a)^2}$
	$k^2\,a^k$	$\dfrac{az\,(z+a)}{(z-a)^3}$
	$(k+1)\,a^k$	$\dfrac{z^2}{(z-a)^2}$
	$\dfrac{(k+1)\,(k+2)}{2!}\,a^k$	$\dfrac{z^3}{(z-a)^3}$
	$\dfrac{(k+1)\,(k+2)\,(k+3)}{3!}\,a^k$	$\dfrac{z^4}{(z-a)^4}$
	$\dfrac{a^k}{k!}$	$e^{az^{-1}}$
t	kT	$\dfrac{Tz}{(z-1)^2}$
t^2	$(kT)^2$	$\dfrac{T^2 z\,(z+1)}{(z-1)^3}$
e^{-at}	e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
te^{-aT}	kTe^{-akT}	$\dfrac{z\,T\,e^{-aT}}{(z-e^{-aT})^2}$
$\sin \omega t$	$\sin \omega kT$	$\dfrac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
$\cos \omega t$	$\cos \omega kT$	$\dfrac{z\,(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$

LINEAR DISCRETE-TIME SYSTEMS

A discrete time system is a device or algorithm that operates on a discrete-time signal called the input a excitation, according to some well-defined rule, to produce another discrete time signal called o/p or response of the system.



$$c(k) = H[r(k)]$$

H denotes the transformation.

A linear system obeys principle of superposition.

$h(k) = H[\delta(k)]$ where $\delta(k)$ = unit impulse

and $h(k)$ = impulse response of the system. It is also called weighting sequence.

TRANSFER FUNCTION OF LDS SYSTEM (PULSE TRANSFER FUNCTION)

Transfer function of LDS system is given by Z-transform of its impulse response. It is also called pulse transfer function or

Let $h(k)$ = Impulse response

$$Z\{h(k)\} = H(z)$$

Then $TF = H(z) = \dfrac{C(z)}{R(z)}$

$C(z) = Z\ c(k)$ where $c(k)$ is the response

$R(z) = Z\ r(k)$ where $r(k)$ is the excitation

Proof:

$$C(z) = Z\{c(k)\} = \sum_{k=0}^{\infty} c(k) z^{-k}$$

$$c(k) = \sum_{m=0}^{\infty} x(m) h(k-m)$$

$$c(k) = H[x(k)]$$

$$x(k) = \sum_{m=0}^{\infty} x(m) \delta(k-m)$$

$\delta(k-m)$ = delayed unit impulses

$$c(k) = H\left[\sum_{m=0}^{\infty} x(m)\, \delta(k-m)\right]$$

$$= \sum_{m=0}^{\infty} x(m) H[\delta(k-m)]$$

$$= \sum_{m=0}^{\infty} x(m)\, h(k-m)$$

$$C(z) = \sum_{k=0}^{\infty}\left[\sum_{m=0}^{\infty} x(m) h(k-m)\right] z^{-k}$$

Interchanging order of summation

$$C(z) = \sum_{m=0}^{\infty} x(m) \sum_{k=0}^{\infty} h(k-m) z^{-k}$$

Let $p = (k-m)$, when $k=0$, $p=-m$

$k=\infty$, $p=\infty$

$k = p+m$

$$\therefore C(z) = \sum_{m=0}^{\infty} x(m) \sum_{p=-m}^{\infty} h(p) z^{-p+m}$$

$$= \sum_{m=0}^{\infty} x(m) \sum_{p=0}^{\infty} h(p) z^{-p} z^{-m} \quad (h(p)=0 \text{ for } p<0)$$

$$= \sum_{m=0}^{\infty} x(m) z^{-m} \sum_{p=0}^{\infty} h(p) z^{-p}$$

$$C(z) = R(z) H(z)$$

Transfer function of LDS system $= \dfrac{C(z)}{R(z)}$ (Ratio of z transform of o/p of a s/m to z transform of i/p to s/m when zero i/p ...)

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Additional Answer Sheet
(Two Writing Pages)

→ The input-output relation of a sampled data system is described by the equation (4)

$$c(k+2) + 3c(k+1) + 4c(k) = r(k+1) - r(k)$$

Determine the z-transfer function. Also obtain the weighting sequence of the system.

Ans: Let $R(z) = Z\{r(k)\}$ and $C(z) = Z\{c(k)\}$

$$Z\{c(k+m)\} = z^m C(z)$$

$$\therefore z^2C(z) + 3zC(z) + 4C(z) = zR(z) - R(z)$$

$$\Rightarrow \frac{C(z)}{R(z)} = \frac{z-1}{z^2+3z+4} = H(z)$$

weighting sequence $= h(k)$

ie $Z^{-1}\{H(z)\}$

$$H(z) = \frac{z-1}{\left(z+\frac{3}{2}+j\frac{\sqrt{7}}{2}\right)\left(z+\frac{3}{2}-j\frac{\sqrt{7}}{2}\right)}$$

$$= \frac{A}{z+\frac{3}{2}+j\frac{\sqrt{7}}{2}} + \frac{A^*}{z+\frac{3}{2}-j\frac{\sqrt{7}}{2}}$$

$$A = \frac{1}{2} - j\frac{5}{2\sqrt{7}}$$

$$A^* = \frac{1}{2} + j\frac{5}{2\sqrt{7}}$$

$$\therefore H(z) = \frac{\frac{1}{2}-j\frac{5}{2\sqrt{7}}}{z+\frac{3}{2}+j\frac{\sqrt{7}}{2}} + \frac{\frac{1}{2}+j\frac{5}{2\sqrt{7}}}{z+\frac{3}{2}-j\frac{\sqrt{7}}{2}}$$

$\left(Z(a^k) = \frac{z}{z-a}\right)$

$Z(a^{k-1}) = z^{-1}\frac{z}{z-a}$

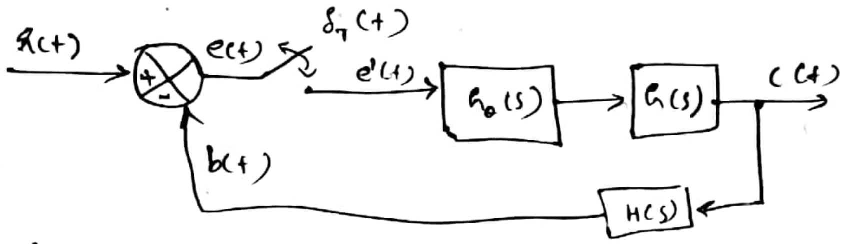
$$= \left(\frac{1}{2}-j\frac{5}{2\sqrt{7}}\right)z^{-1} \times \frac{z}{z-\left(-\frac{3}{2}-j\frac{\sqrt{7}}{2}\right)} + \left(\frac{1}{2}+j\frac{5}{2\sqrt{7}}\right)z^{-1} \times \frac{z}{z-\left(-\frac{3}{2}+j\frac{\sqrt{7}}{2}\right)}$$

$$\therefore h(k) = \left(\frac{1}{2}-j\frac{5}{2\sqrt{7}}\right)\left(-\frac{3}{2}-j\frac{\sqrt{7}}{2}\right)^{k-1}u(k-1) + \left(\frac{1}{2}+j\frac{5}{2\sqrt{7}}\right)\left(-\frac{3}{2}+j\frac{\sqrt{7}}{2}\right)^{k-1}u(k-1)$$

STABILITY ANALYSIS OF SAMPLED DATA CONTROL SYSTEMS

The sampled data control system is stable if all the poles of the Z-transfer function of the system lies inside the unit circle in Z-plane.

Poles of transfer function are given by the roots of the characteristic equation.



$$H(z) = \frac{C(z)}{R(z)} = \frac{Z\{G_0(s)\,G(s)\}}{1 + Z\{G_0(s)\,G(s)\,H(s)\}}$$

The characteristic equation is,

$$1 + Z\{G_0(s)\,G(s)\,H(s)\} = 0$$

JURY'S STABILITY TEST

This test is used to determine whether the roots of the characteristic polynomial lie within a unit circle or not.

Let $F(z)$ be the n^{th} order characteristic polynomial of a sampled data control system.

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots a_2 z^2 + az + a_0 = 0$$

Where $a_n > 0$ and $a_0, a_1, a_2 \dots a_n$ are constant coeffi

Necessary condition

$F(1) > 0$ and $(-1)^n F(-1) > 0$

Sufficient condition

Prepare a table using the coefficients of the characteristic polynomial $F(z)$. Table consist of $(2n-3)$ rows, where n is the order of characteristic equ

Row	z^0	z^1	z^2	...	z^{n-k}	...	z^{n-2}	z^{n-1}	z^n
1	a_0	a_1	a_2	...	a_{n-k}	...	a_{n-2}	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	...	a_k	...	a_2	a_1	a_0
3	b_0	b_1	b_2	...		...	b_{n-2}	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	...		...	b_1	b_0	
5	c_0	c_1	c_2	...		...	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	...		...	c_0		
.	.	.	.						
.	.	.	.						
$2n-5$	s_0	s_1	s_2	s_3					
$2n-4$	s_3	s_2	s_1	s_0					
$2n-3$	r_0	r_1	r_2						

The elements of row 1 are formed using the coefficients of characteristics polynomial and the row 2 is formed by arranging the elements of row 1 in the reverse order.

k^{th} element of row 3, $b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$

i.e. $b_0 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}$, $b_1 = \begin{vmatrix} a_0 & a_{n-1} \\ a_n & a_1 \end{vmatrix}$ .. and so on

4^{th} row is obtained by reversing 3^{rd} row.

5^{th} row $\longrightarrow$ $c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$

$c_0 = \begin{vmatrix} b_0 & b_{n-1} \\ b_{n-1} & b_0 \end{vmatrix}$, $c_1 = \begin{vmatrix} b_0 & b_{n-2} \\ b_{n-1} & b_1 \end{vmatrix}$ ...

6^{th} row is obtained by reversing 5^{th} row and continue.

The 1^{st} column elements of the table are used to check the following $(n-1)$ conditions.
These $(n-1)$ conditions are the sufficient conditions for stability

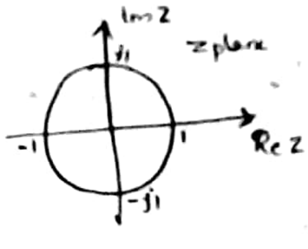
$$\left.\begin{array}{l} |a_0| < |a_n| \\ |b_0| > |b_{n-1}| \\ |c_0| > |c_{n-2}| \\ \vdots \\ |r_0| > |r_2| \end{array}\right\} \quad n-1 \text{ condition}$$

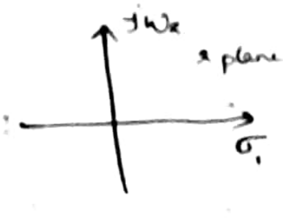
If the necessary and sufficient conditions are satisfied then all the poles of the system lies inside the unit circle in Z-plane and so the system is stable.

STABILITY ANALYSIS USING BILINEAR TRANSFORMATION

The bilinear transformation maps the interior of unit circle in the Z plane into the left half of the s plane.

In this transformation $s = \frac{Z-1}{Z+1}$, or $Z = \frac{1+s}{1-s}$.





Consider the characteristic equation of the system in Z-plane

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots a_1 z + a_0 = 0 \quad ; \; a_n > 0$$

Using bilinear transformation

$$a_n \left(\frac{1+s}{1-s}\right)^n + a_{n-1}\left(\frac{1+s}{1-s}\right)^{n-1} + \dots + a_1\left(\frac{1+s}{1-s}\right) + a_0 = 0$$

It can be simplified as $b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0 = 0$

For the stability of the system, the roots of new characteristic

equation should lie on the left half of s-plane.
The routh stability criterion can be applied to the new characteristic equation.

Necessary condition

coefficients of characteristic (transformed) equation should be +ve.

Sufficient condition

There should not be any sign change in the elements of 1st column of routh array.

The system is stable if both the necessary and sufficient conditions are satisfied.

s^n b_n b_{n-2} b_{n-4} ... row 1

s^{n-1} b_{n-1} b_{n-3} b_{n-5} ... row 2

s^{n-2} c_1 c_2 c_3 ...

s^{n-3} d_1 d_2 d_3 ...

$\vdots$

s^0 ... row (n+1)

$$c_1 = \frac{b_{n-1} \times b_{n-2} - b_n \times b_{n-3}}{b_{n-1}}$$

$$c_2 = \frac{b_{n-1} \times b_{n-4} - b_n \times b_{n-5}}{b_{n-1}}$$

$$d_1 = \frac{c_1 \times b_{n-3} - b_{n-1} c_2}{c_1}$$

